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A similarity solution in order to solve the governing equations of laminar separated fluids with a flat plate

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ABSTRACT

In this paper, the development of local and non-local similarity solutions for laminar flow and heat transfer between two separated fluids is described. This paper focuses on the extension of similarity solutions of convective surface boundary condition. This new case represents two adjacent fluids separated by a flat plate, and moving parallel to each other, where the convective heat transfer coefficient of the fluid heating the plate on its lower surface is proportional to $x^{-1/2}$. Numerical and analytical solutions are available to solve this boundary value problem. Within the first case, shooting method is applied; furthermore, Runge–Kutta fourth order method is used for integration over the whole boundary layer. Numerical solutions of the resulting similarity energy equation are represented for various Prandtl numbers and a range of values of the parameter characterizing the hot fluid convection process. In addition, analytical exact series solutions are provided for all different Prandtl numbers, although for cold fluids with low Prandtl numbers, a compact solution is also obtained. Finally, an appropriate range of Prandtl number is obtained in which compact and exact solution have a good agreement.

1. Introduction

Many researches related to the laminar hydrodynamic and thermal boundary layer, have been done. One of the most important and popular similarity solutions is Blasius equation [1]. Blasius similarity solution gives the velocity distribution in the hydrodynamic boundary layer by reducing momentum equation to an ordinary differential equation. Many researchers have considered different boundary conditions such as constant surface temperature [2], constant heat flux at the plate [3], different variation of heat flux or surface temperature [4,5], and etc. to give different similarity solutions. Aziz [6] has demonstrated that a similarity solution is possible for a convective boundary condition at the plate, where the convective heat transfer of the fluid heating the plate on its lower surface is proportional to $x^{-1/2}$. Cortell [7] has analyzed the effects of thermal radiation on the laminar boundary layer about a flat plate in a uniform fluid stream (Blasius flow), and about a moving plate in a quiescent ambient fluid (Sakiadis flow) both under a convective surface boundary condition.

In this paper, first, we focus on numerical solutions of the resulting thermal similarity equation for laminar flow and heat transfer between two separated fluids for various Prandtl numbers and a range of values characterizing the hot fluid convection process. Then, analytical solutions will be presented which consist of two approaches. Exact series solution will be provided which covers all ranges of Prandtl numbers and different cold or hot fluids; in addition, by means of the error function, a compact relation will be suggested for low Prandtl-number cold fluids, in order to evaluate the rate of the heat transfer between hot and cold fluid through the flat plate.

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2. Mathematical analysis

The major convection parameters may be obtained by solving the appropriate form of the boundary layer equations. We consider the problem of hydrodynamic and thermal boundary layer flows over a flat plate in a stream of the cold fluid at temperature T_∞ moving over the top surface of the plate with a uniform velocity U_∞. Assuming steady, incompressible, laminar flow with constant fluid properties and negligible viscous dissipation, and recognizing that $\frac{dp}{dx} = 0$, the boundary layer equations can be written as:

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2} \tag{2}$$

Energy:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where u and v are the x (along the plate) and the y (normal to the plate) components of the velocities, respectively, T is the temperature, υ is the kinematic viscosity of the fluid, and α is the thermal diffusivity of the fluid.

The appropriate hydrodynamic boundary conditions are:

$$u(x,0) = v(x,0) = 0 \tag{4}$$

$$T(x,\infty) = T_\infty \tag{5}$$

As mentioned before, the bottom surface of the plate is heated by convection from a hot fluid at temperature T_f which provides a heat transfer coefficient h_f. The boundary conditions at the plate surface and far into the cold fluid may be written as:

$$-k\frac{\partial T}{\partial y}(x,0) = h_f[T_f - T(x,0)] \tag{6}$$

$$T(x,\infty) = T_\infty \tag{7}$$

Now, an independent η variable and a dependent variable f, in terms of the stream function, are introduced as:

$$\eta = y\left(\frac{U_\infty}{\upsilon x}\right)^{1/2} \tag{8}$$

$$f(\eta) = \frac{\Psi}{U_\infty\sqrt{\left(\frac{\upsilon x}{U_\infty}\right)}} \tag{9}$$

Application of these variables simplifies matters by reducing the partial differential equations, (2) and (3), to the ordinary differential equations.

Similarly defining a dimensionless temperature θ as:

$$\theta = \frac{T - T_\infty}{T_f - T_\infty} \tag{10}$$

Eqs. (1)–(3) reduce to:

$$2f''' + ff'' = 0 \tag{11}$$

$$\theta'' + 1/2Prf\theta' = 0 \tag{12}$$

Here the primes denote differentiation of f with respect to η.

The boundary conditions in terms of the similarity variables are:

$$f(0) = f'(0) = 0 \tag{13}$$

$$f'(\infty) = 1 \tag{14}$$

$$\theta'(0) - \alpha[1 - \theta(0)] \tag{15}$$

$$\theta(\infty) = 0 \tag{16}$$

where

$$a = \frac{h_f}{k}\sqrt{vx/U_\infty} \tag{17}$$

For the energy equation to have a similarity solution, the quantity α must be a constant and not a function of x. This condition can be met if the heat transfer coefficient h_f is proportional to $x^{-1/2}$. Now, assume that:

$$h_f = cx^{-1/2} \tag{18}$$

After defining α solutions of Eqs. (11)–(16) yield the similarity solutions. With respect to α defined by Eq. (17), the solutions generated are the local similarity solutions.

3. Numerical solutions

Eqs. (11), (13), and (14) constitute the classical Blasius problem which is studied by numerous researchers such as Cortrell [8].

In this paper, we focus on the solution of the energy equation. In order to approach the goal, the FORTRAN software was used. The asymptotic boundary conditions equations (14) and (15) $\eta = \infty$ at were replaced by those at $\eta = 8$ in accordance with standard practice in the boundary layer analysis. Runge–Kutta fourth order method and shooting method were used to solve the boundary value problem.

Table 1
Numerical solutions for temperature variation on the flat plate.

α	$\theta(0)$	$-\theta'(0)$
$Pr = 0.05$		
0.05	0.25969	0.03701566
0.1	0.41230	0.05876962
0.2	0.58387	0.08322519
0.4	0.73727	0.10509216
0.6	0.80804	0.11517814
0.8	0.84877	0.12098391
1	0.87524	0.12475713
5	0.97228	0.13858999
10	0.98595	0.14053660
20	0.99292	0.14153102
$Pr = 0.1$		
0.05	0.23879	0.03806060
0.1	0.38552	0.06144813
0.2	0.55650	0.08870054
0.4	0.71506	0.11397451
0.6	0.79010	0.12593739
0.8	0.83386	0.13291096
1	0.86252	0.13747868
5	0.96911	0.15446949
10	0.98431	0.15689044
20	0.99209	0.15813069
$Pr = 0.5$		
0.05	0.16153	0.04192336
0.1	0.27814	0.07218628
0.2	0.43522	0.11295543
0.4	0.60649	0.15740470
0.6	0.69805	0.18116870
0.8	0.75505	0.19596121
1	0.79394	0.20605591
5	0.95065	0.24672765
10	0.97470	0.25297653
20	0.98719	0.25621367
$Pr = 1$		
0.05	0.13087	0.04345652
0.1	0.23145	0.07685508
0.2	0.37590	0.12482052
0.4	0.54640	0.18143867
0.6	0.64374	0.21375873
0.8	0.70668	0.23465889
1	0.75072	0.24928301
5	0.93772	0.31138042
10	0.96786	0.32138872
20	0.98367	0.32664705

Table 1 shows the results of the problem for fixed Prandtl numbers of 0.05, 0.1, 0.5, and 1. For each Prandtl number, both $\theta(0)$ and $\theta'(0)$ increase as α increases.

According to the results, as $\alpha \to \infty$, the solution approaches the classical solution for the constant surface temperature. This can be seen from the boundary condition equation (15) which reduces to $\theta(0) = 1$ as $\alpha \to \infty$.

By obtaining $\theta(0)$ and $\theta'(0)$ the Nusselt number and the total heat transfer rate can be gained which will be discussed later.

It was observed that numerical results of this study are in a good agreement with [6].

4. Analytical solutions

4.1. Case. 1

In order to represent a simple form of analytical solution we refer again to Eq. (12). This equation can be written in this form:

$$\frac{\theta''(\eta)}{\theta'(\eta)} = -\frac{1}{2}Prf \tag{19}$$

From Blasius equation (11):

$$f = -2\frac{f'''}{f''} \tag{20}$$

Substitution the right hand side of Eq. (19) into Eq. (20), and some simplifications yield:

$$\frac{\theta'(\eta)}{\theta'(\eta)} = Pr\frac{f'''(\eta)}{f''(\eta)} \tag{21}$$

Integrating both sides of Eq. (21) from 0 to η gives:

$$\frac{\theta''(\eta)}{\theta'(0)} = \left[\frac{f'''(\eta)}{f''(0)}\right]^{pr} \tag{22}$$

Applying boundary condition for $\theta'(0)$:

$$\theta'(\eta) = \frac{-a(1-\theta(0))}{f''(0)^{pr}}f''(\eta)^{pr} \tag{23}$$

Again, integrating both sides of Eq. (23) from 0 to η gives

$$\theta'(\eta) - \theta(0) = \frac{-a(1-\theta(0))}{f''(0)^{pr}}\int_0^\eta f''(\xi)^{pr}\,d\xi \tag{24}$$

where, ξ is a dummy variable of the integration. Noting $\frac{-a(1-\theta(0))}{f''(0)^{pr}}$ is constant, so this term comes out from the integral.

At this point two different approaches are available, in the first approach we do not find $\theta(\eta)$ directly; the favor quantity is $\theta(0)$, and we can find it simply in compact form.

Second approach is analytical solution based on Blasius series solution [1], and next, an approximate analytic solution of Blasius problem [8], based on Pade approximation and numerical results, were applied to this traditional solution.

4.2. First approach

Second boundary condition is applied, $\theta(\infty) = 0$ so by means of $\eta \to \infty$ in both sides of Eq. (24).

$$\theta(\infty) - \theta(0) = \frac{-a[1-\theta(0)]}{f''(0)^{Pr}}\int_0^\infty f''(\xi)^{Pr}\,d\xi \tag{25}$$

After some simplification, it gives:

$$\theta(0) = 1 - \frac{1}{1 + \frac{a}{f''(0)^{Pr}}\int_0^\infty f''(\xi)^{Pr}\,d\xi} \tag{26}$$

This is, in fact, the general relation of $\theta(0)$, for any arbitrary values of α and Pr. In the special case, when the Prandtl number of the fluid is unity, Eq. (26) gives:

$$\theta(0) = 1 - \frac{1}{1 + \frac{a}{f''(0)}[f'(\infty) - f'(0)]} \tag{27}$$

Applying boundary conditions of Blasius equation, gives:

$$\theta(0) = 1 - \frac{f''(0)}{a + f''(0)} \tag{28}$$

$f''(0)$ may be found by Howarth correct to five decimals position, numerically as $f''(0) = 0.33206$, this relation can be compared with numerical results for $Pr = 1$ and different values of α in Table 2 In the above table, $\theta'(0)$ was found simply by (15).

According to numerical and analytical solutions it can be observed that these results agree up to four places of decimal.

Now, the temperature distribution on the flat plate for various values of parameter α will be shown which is characterizing the convection process of the hot fluid heating the plate on its lower surface.

Solutions of the similarity solution for numerical approaches for a fixed Prandtl number of 0.72 can be met in [6]. As it can be met, $\theta(0) \to 1$ as α increases, and the solution approaches the classical constant-temperature flat plate problem $(T(x,0) = T_f)$. Finally, it may be realized that as $\alpha \to \infty$ then both $\theta(0)$ and $-\theta'(0)$ increase, so the heat flux and heat transfer rate increase seriously. Thus this case is appropriate to enhance the heat flux and heat transfer between two separated fluids.

Dimensionless temperature versus Prandtl number has been shown in Fig. 1. According to this figure by decreasing the Prandtl number of the cold fluid i.e. highly conductive fluids, the temperature of the plate approaches the hot-fluid temperature (T_f) for smaller magnitudes of parameter α. Note that for fixed cold fluid properties and fixed free stream velocity, α at any location x is proportional to the heat transfer coefficient associated with the hot fluid (h_f), so small values of α mean lower heat transfer rate between the hot and cold fluid.

Table 2
Analytical solutions for variation of temperature on the flat plate.

α	$\theta(0)$	$-\theta'(0)$
$Pr = 0.05$		
0.05	0.25969	0.03701566
0.1	0.41230	0.05876962
0.2	0.58387	0.08322519
0.4	0.73727	0.10509216
0.6	0.80804	0.11517814
0.8	0.84877	0.12098391
1	0.87524	0.12475713
5	0.97228	0.13858999
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$Pr = 0.1$		
0.05	0.23879	0.03806060
0.1	0.38552	0.06144813
0.2	0.55650	0.08870054
0.4	0.71506	0.11397451
0.6	0.79010	0.12593739
0.8	0.83386	0.13291096
1	0.86252	0.13747868
5	0.96911	0.15446949
10	0.98431	0.15689044
20	0.99209	0.15813069
$Pr = 0.5$		
0.05	0.16153	0.04192336
0.1	0.27814	0.07218628
0.2	0.43522	0.11295543
0.4	0.60649	0.15740470
0.6	0.69805	0.18116870
0.8	0.75505	0.19596121
1	0.79394	0.20605591
5	0.95065	0.24672765
10	0.97470	0.25297653
20	0.98719	0.25621367
$Pr = 1$		
0.05	0.13087	0.04345652
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0.2	0.37590	0.12482052
0.4	0.54640	0.18143867
0.6	0.64374	0.21375873
0.8	0.70668	0.23465889
1	0.75072	0.24928301
5	0.93772	0.31138042
10	0.96786	0.32138872
20	0.98367	0.32664705

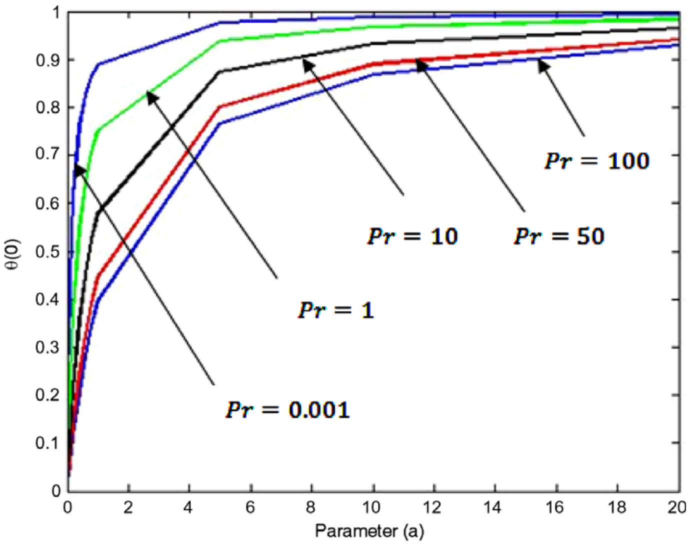

Fig. 1. Dimensionless temperature versus parameter α for various fluids with different Prandtl numbers.

4.3. Second approach

In the second approach, Blasius series solution [1] is used to compute the right hand side of Eq. (26). From Blasius solution:

$$f(\eta) = \sum_{k=0}^{\infty} \left(\frac{-1}{2}\right)^k \frac{A_k f''(0)^{k+1}}{(3k+2)!} \eta^{3k+2} \tag{29}$$

where $A_0 = A_1 = 1$ and

$$A_k = \sum_{0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1}, \quad k \geqslant 2 \tag{30}$$

So

$$f''(\eta) = \sum_{0}^{\infty} \left(\frac{-1}{2}\right)^k \frac{A_k f''(0)^{k+1}}{(3k)!} \eta^{3k} \tag{31}$$

With substituting Eq. (31) into (27) the general solution of $\theta(0)$ can be found.

$$\theta(0) = 1 - \frac{1}{1 + \frac{\alpha}{f''(0)^{Pr}} \int_0^{\infty} \left(\sum_{k=0}^{\infty} \left(\frac{-1}{2}\right)^k \frac{A_k f''(0)^{k+1}}{(3k)!} \eta^{3k} \right)^{Pr} d\eta} \tag{32}$$

In spite of the presence of $(3k+1)!$ in the denominator of Blasius series solution, the above series converges only within a finite integral $[0, \eta_{max}]$ where, $\eta_{max} = \frac{1.8894}{f''(0)}$, but [9] has proposed approximate analytical solution for $f'(\eta)$ as:

$$f'(\eta) = \frac{0.332057_{\eta} + 0.00059069^4_{\eta} + 0.00000288\eta^5 \exp(\eta^2/4 - 1))}{1 + 0.00869674^3_{\eta} + 0.000002888^5_{\eta} \exp(\eta^2/4 - 1)} \tag{33}$$

The above suggestion will hold in the whole domain $[0, \infty)$ for η.

We can take derivative from Eq. (33) and put it to Eq. (27) to take the integral analytically, but it is very complicated.

4.4. Case. 2

It is still better to develop an analytical replacement for the highly conductive fluids. For this type of fluids the velocity boundary layer is much thinner than the thermal layer. Therefore, in this limit it is permissible to set $f' = 1$ in the region occupied by the thermal boundary layer. Differentiating equation (12) once:

$$\frac{d}{d\eta}\left(\frac{\theta''}{\theta'}\right) = -\frac{Pr}{2} f' \tag{34}$$

This equation leads to an explicit solution for $\theta(\eta)$ as $Pr \to 0$.

$$d\left(\frac{\theta''}{\theta'}\right) = -\frac{1}{2}Prf'\,d\eta \tag{35}$$

Integrating Eq. (35)

$$\frac{\theta''(\eta)}{\theta'(\eta)} - \frac{\theta''(0)}{\theta'(0)} = -\frac{Pr}{2}\eta \tag{36}$$

According to Eq. (12)

$$\theta''(0) + \frac{1}{2}Prf(0)\theta'(0) = 0 \tag{37}$$

So,

$$\theta''(0) = 0 \tag{38}$$

Substituting Eq. (38) into (36), and integrating once

$$\theta'(\eta) = \theta'(0)\exp\left(-\frac{Pr}{4}\eta^2\right) \tag{39}$$

According to the boundary conditions

$$\theta'(0) = -a[1-\theta(0)] \tag{40}$$

By substituting Eq. (40) into (39), and integrating

$$\theta(\eta) - \theta(0) = -a[1-\theta(0)]\sqrt{\frac{\pi}{Pr}}\mathrm{erf}(x) \tag{41}$$

where, x is a function of and defined as

$$x = \sqrt{\frac{Pr}{2}}\eta \tag{42}$$

Now, if $\eta \to \infty$ then Eq. (41) reduces to

$$\theta(0) = \frac{a\sqrt{\frac{\pi}{Pr}}\mathrm{erf}\left(\frac{\sqrt{Pr}}{2}\eta\right)}{1 + a\sqrt{\pi/Pr}\,\mathrm{erf}\left(\frac{\sqrt{Pr}}{2}\eta\right)} \tag{43}$$

In this case, $Pr \to 0$, and for high numbers of parameter α, $\theta(0) \to 1$ and $\theta'(0) \to 0$, so heat transfer reduces seriously; therefore, this condition renders liquid metals unattractive as coolants in applications requiring high heat transfer rate.

The Nusselt number and the total heat transfer rate q can be expressed in terms of $\theta'(0)$ as follows:

$$Nu = -\left(\frac{T_f - T_\infty}{T - T_\infty}\right)\sqrt{\frac{U_\infty x}{v}}\theta'(0) \tag{44}$$

$$q = -2kW(T_f - T_\infty)\sqrt{\left(\frac{U_\infty L}{v}\right)}\theta'(0) \tag{45}$$

where, α is a constant.

$$q = -kWL(T_f - T_\infty)\sqrt{\frac{U_\infty}{v}}\int_0^L x^{-\frac{1}{2}}\theta'(0)\,dx \tag{46}$$

where, α is a function of x.

In Eqs. (44)–(46), L is the plate length and W is the plate width. It should be noted for the case (α = a function of x), $\theta'(0)$ depends on x. Thus integration over the entire plate is necessary to obtain the total heat transfer rate. In addition, after gaining the approximate compact relation, it is compared with exact solution. Figs. 2–4 represent this comparison for various Prandtl numbers: 0.001, 0.01, 0.05, 0.1, 0.5 and 1. According to these figures the discrepancy between exact and approximate solutions decreases as parameter α increases; in this case $\theta(0)$ approaches 1 as α increases mutually according to Eq. (15), $\theta'(0)$ approaches 0, so heat transfer between separated fluids reduces seriously. Note that when Pr number increases the discrepancy between the exact and approximate solution (compact formula) increases, as we expected; as a matter of fact, this happens because the compact relation is exact for the case of $Pr \to 0$. In order to find appropriate range of Pr number, exact and approximate solutions are compared for $0.001 \leqslant Pr \leqslant 1$.

According to these figures, it may be concluded that the compact solution can be used for $Pr \leqslant 0.1$. In this case the discrepancy between the exact and approximate solutions is very small.

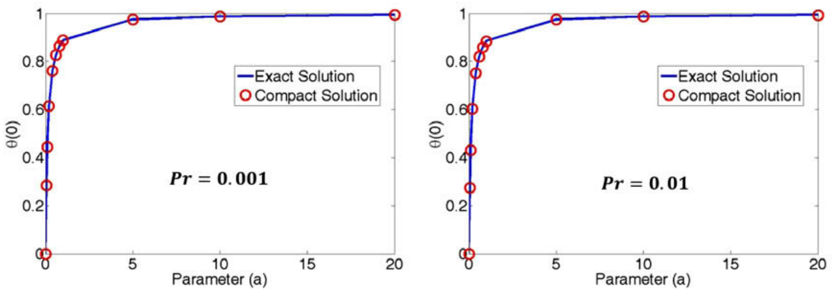

Fig. 2. Comparison of exact and approximate solutions for (a): $Pr = 0.001$ and (b): $Pr = 0.01$.

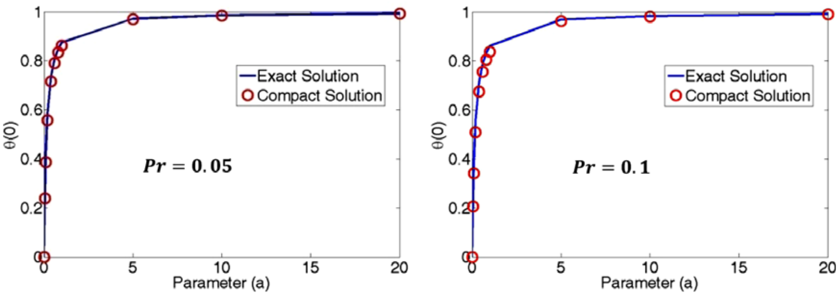

Fig. 3. Comparison of exact and approximate solutions for (a): $Pr = 0.05$ and (b): $Pr = 0.1$.

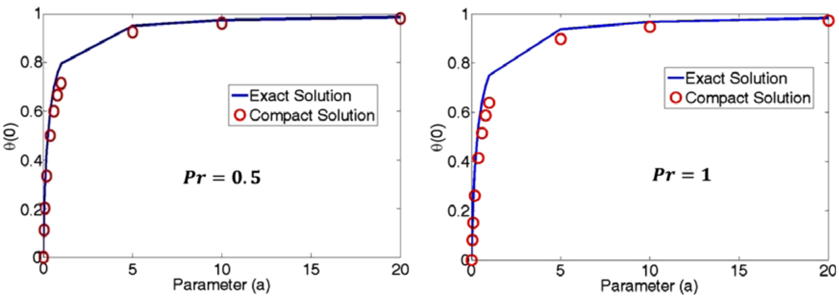

Fig. 4. Comparison of exact and approximate solutions for (a): $Pr = 0.5$ and (b): $Pr = 1$.

5. Conclusion

A similarity solution for the laminar flow and heat transfer between two separated fluids is possible if the convective heat transfer of the fluid heating the plate on its lower surface is proportional to $x^{-1/2}$. Numerical solutions of the resulting thermal similarity equation reported for four representative Prandtl numbers of 0.05, 0.1, 0.5, and 1. Then, analytical solutions expressed which consisted of two cases. Case. 1 solved the problem by means of series solution which is exact for all ranges of Prandtl numbers. Case. 2 represented compact formula for highly conductive cold fluids. Relations to evaluate the Nusselt number and total heat transfer rate were suggested; in addition, exact and approximate solutions were compared in order to obtain the discrepancy between results. Finally, appropriate range of Prandtl number was evaluated for proper application of the approximate solution.

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